**Partial Fractions Decomposition**

**Partial fractions decomposition is only used for rational functions (polynomials divided by polynomials) in which the degree of the numerator is less than the degree of the denominator.**

Partial fractions decomposition is used in various contexts in Math 1B, 1C and 2A, and other advanced math.
The process of decomposition may be slightly modified for its particular use in each class,
but the basic principle remains the same.

## To find the denominators and the forms of the numerators of the partial fractions:

1. Completely factor the denominator into a product of linear factors ( $ax+b$ ) and/or quadratic factors ( $ax^2+bx+c$ ) with negative discriminants ( $b^2-4ac<0$ ).

   **Any factors of degree 3 or higher can still be factored into linear and quadratic factors.**
   The linear factors correspond to the real (integer, rational and irrational) roots of the polynomial.
   The quadratic factors correspond to the pairs of complex conjugate roots of the polynomial.

2. Each linear factor ( $ax+b$ ) gives rise to a partial fraction of the form $\dfrac{A}{ax+b}$,
   where $A$ must be a constant.

   If a particular linear factor has multiplicity $n$ (ie. it appears as $(ax+b)^n$ in the factoring),
   that factor actually gives rise to a series of $n$ partial fractions
   with the positive integer powers (1 to $n$) of $ax+b$ as their denominators,
   and constants as their numerators.
   ie. $\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\dfrac{A_3}{(ax+b)^3}+\cdots+\dfrac{A_n}{(ax+b)^n}$

3. Each quadratic factor ( $ax^2+bx+c$ ) gives rise to a partial fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$,
   where $A$ and $B$ must be constants

   If a particular quadratic factor has multiplicity $n$ (ie. it appears as $(ax^2+bx+c)^n$ in the factoring),
   that factor actually gives rise to a series of partial fractions
   with the positive integer powers (1 to $n$) of $ax^2+bx+c$ as their denominators,
   and linear functions as their numerators.
   ie. $\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\dfrac{A_3x+B_3}{(ax^2+bx+c)^3}+\cdots+\dfrac{A_nx+B_n}{(ax^2+bx+c)^n}$

NOTE:
The format of the linear function in the numerators may be represented differently in each class.
For example, in both Math 1B and 2A, we complete the square in the denominator,
writing it in the form $a[(x+h)^2+k^2]$.
In Math 1B, we use partial fractions decomposition to find antiderivatives,
so we write the numerator in the form $A(2(x+h))+B(k)$.
In Math 2A, we use partial fractions decomposition to find inverse Laplace transforms,
so we write the numerator in the form $A(x+h)+B(k)$.
These alternate ways of writing the numerator are designed to reduce the amount of work required
in the later steps of finding antiderivatives and inverse Laplace transforms.

eg. $$\frac{N(x)}{(10x-9)(8x+7)^4(6x^2-5x+4)(3x^2+2x+1)^3}$$

The degree of the denominator is $1+(1\times 4)+2+(2\times 3)=13$

The numerator $N(x)$ must be a polynomial of degree $\le 12$

$$= \frac{A}{10x-9}$$

$10x-9$ is linear, so the numerator must be a constant

$$+\frac{B_1}{8x+7}+\frac{B_2}{(8x+7)^2}+\frac{B_3}{(8x+7)^3}+\frac{B_4}{(8x+7)^4}$$

$8x+7$ is linear, so the numerators must be constants

Its multiplicity is $4$, so there are $4$ fractions (powers $1$ to $4$)

$$+\frac{Cx+D}{6x^2-5x+4}$$

$6x^2-5x+4$ is quadratic, so the numerator must be linear

NOTE: The discriminant is $(-5)^2-4(6)(4)<0$

$$+\frac{E_1x+F_1}{3x^2+2x+1}+\frac{E_2x+F_2}{(3x^2+2x+1)^2}+\frac{E_3x+F_3}{(3x^2+2x+1)^3}$$

$3x^2+2x+1$ is quadratic, so the numerators must be linear
Its multiplicity is $3$, so there are $3$ fractions (powers $1$ to $3$)
NOTE: The discriminant is $(2)^2-4(3)(1)<0$

**You must get the correct form for the partial fractions**
**before you attempt to find the unknown constants in the numerators,**
**especially if you use any shortcuts for finding the constants (eg. the method shown in lecture).**
**If not, the shortcuts will give you "answers", but they will be wrong.**

Practice writing the forms of the partial fractions decompositions of the following rational expressions.
**Do NOT find the unknown constants in the numerators.**
What is the maximum degree of each numerator for which partial fractions decomposition will work ?

[1] $$\frac{N(x)}{x^3(3x^2-2x+4)^2}$$

[2] $$\frac{N(x)}{(x^2+5x+6)(x^2-4x-12)}$$

[3] $$\frac{N(x)}{x^4-8x^2-9}$$

# To find the unknown constants in the numerators of the partial fractions:

A. Multiply both the original fraction and the decomposed sum of the partial fractions by the denominator of the original fraction, and set the results equal to each other.

B. Find the constants by either comparing the constant terms and the coefficients of the powers of $x, x^2, x^3, \ldots, x^{d-1}$ in the two expressions resulting from step A (where $d$ is the degree of the denominator), or by substituting appropriate values of $x$ into the two expressions (as shown in lecture).

C. Sanity check your partial fractions decomposition by substituting a value for $x$ into the original fraction and your partial fractions, and check that they give the same result.
**The value of $x$ should NOT be 0, ±1 nor any value you substituted for $x$ to find the unknown constants.**

eg. Perform partial fractions decomposition on $\dfrac{29-x}{(x-3)(x^2-12x+40)}$ in preparation for integration

NOTE: $(-12)^2 - 4(1)(40) < 0$ and $x^2 - 12x + 40 = (x-6)^2 + 4$

$$\frac{29-x}{(x-3)(x^2-12x+40)} = \frac{A}{x-3} + \frac{Bx+C}{x^2-12x+40}$$

$$\frac{29-x}{(x-3)(x^2-12x+40)} = \frac{A}{x-3} + \frac{D[2(x-6)] + E(2)}{(x-6)^2 + 2^2}$$

Multiplying both expressions by $(x-3)(x^2-12x+40)$:

$$29 - x = A[(x-6)^2 + 4] + D[2(x-6)](x-3) + E(2)(x-3)$$

$x = 3$: $\quad 29 - 3 = A(13) + D(0) + E(0) \Rightarrow 26 = 13A \Rightarrow A = 2$

$x = 6$: $\quad 29 - 6 = A(4) + D(0) + E(6) \Rightarrow 23 = 8 + 6E \Rightarrow E = \frac{5}{2}$

Coefficient of $x^2$: $\quad 0 = A + 2D \Rightarrow 0 = 2 + 2D \Rightarrow D = -1$

So, $$\frac{29-x}{(x-3)(x^2-12x+40)} = \frac{2}{x-3} + \frac{-[2(x-6)] + \frac{5}{2}(2)}{(x-6)^2 + 4}$$

Sanity check:

$x = 2$: original fraction $\quad \dfrac{27}{(-1)(20)} = -\dfrac{27}{20}$

partial fractions $\quad \dfrac{2}{-1} + \dfrac{13}{20} = \dfrac{-40+13}{20} = -\dfrac{27}{20}$ ✓

The fraction is now ready for integration without additional work in rewriting the fraction.